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Strong Decay to Equilibrium in One-Dimensional Random Spin Systems

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We consider a spin system on a lattice with finite-range, possibly unbounded random interactions. We show that for such systems the Glauber dynamics cannot decay to equilibrium exponentially fast in L_2 even at high temperatures. Additionally, for one-dimensional systems with unbounded random couplings we prove that with probability one the corresponding Glauber dynamics has a fast (subexponential) decay to equilibrium in the uniform norm, provided that the distribution of random couplings satisfies some exponential bound.

KEY WORDS: Stochastic dynamics; random spin systems; logarithmic Sobolev inequality with local coefficients; strong decay to equilibrium.

1. INTRODUCTION

The equilibrium description of lattice spin systems with random interactions at high temperatures is relatively well studied (see refs. 1, 4, 3, 7, and 11, and references given there). In the present paper we make a first attempt to make progress in the study of the corresponding stochastic dynamics.

We consider a spin system on a lattice $\mathbb{Z}^d$ defined by the following interaction energy:

$$U_\Lambda \equiv -\sum_{\mathrm{nn}} J_{\mathrm{ii'}} \sigma_{\mathrm{i}} \sigma_{\mathrm{i'}} \tag{1.1}$$

with nn indicating the summation running over the nearest-neighbor pairs intersecting a finite set $\Lambda \subset \mathbb{Z}^d$ and with spins $\sigma_{\mathrm{i}}, \sigma_{\mathrm{i'}}$ taking values from a finite set $\mathbf{M} \subset \mathbb{R}$. The couplings $J_{\mathrm{i,i'}} \in \mathbb{R}$ are i.i.d. random variables with a

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distribution E, satisfying $E\,|J_{\mathbf{i},\mathbf{i}'}| < \infty$, where we have used a particular case of the notation $EF \equiv \int F\,dE$ for an expectation of a function F with the probability measure E. By $\mathbf{J}$ we denote a configuration of random couplings, i.e., $\mathbf{J} \equiv \{J_{\mathbf{i},\mathbf{i}'}\}_{\mathbf{i},\mathbf{i}' \in \mathbb{Z}^d}$. If the dimension $d = 1$, we will assume that the random couplings are unbounded; otherwise the result of ref. 12 applies.

Let μ_0 be a free measure on $\Omega \equiv \mathbf{M}^{\mathbb{Z}^d}$, defined as the product of uniform measures on $\mathbf{M}$ and, for $\Lambda \in \mathbb{Z}^d$, let μ_0^Λ denote its restriction to the sigma algebra generated by the spins $\sigma_\mathbf{i}$, $\mathbf{i} \in \Lambda$. The expectation of a function f with a probability measure μ on Ω will be denoted by $\mu(f) \equiv \mu f \equiv \int f\,d\mu$. Later we will need the following discrete gradient:

$$\nabla_\mathbf{i} f \equiv \mu_0^\mathbf{i} f - f$$

and the following seminorm:

$$|||f||| \equiv \sum_\mathbf{i} \|\nabla_\mathbf{i} f\|_u$$

with $\|\cdot\|_u$ denoting the supremum norm.

Let $\mathscr{F}$ denote the family of all finite sets in $\mathbb{Z}^d$. For every set $\Lambda \in \mathscr{F}$ we define a finite-volume Gibbs measure μ_Λ^σ, with external conditions given by a configuration $\sigma \in \mathbf{M}^{\mathbb{Z}^d}$, as follows:

$$\mu_\Lambda^\sigma(F) \equiv \delta_\sigma \left(\frac{\mu_0^\Lambda e^{-U_\Lambda} F}{\mu_0^\Lambda e^{-U_\Lambda}} \right) \tag{1.2}$$

where δ_σ denotes the Dirac measure at $\sigma \in \Omega$.

It is known[4,3,7] that (with probability one) the described random spin system has a unique infinite-volume Gibbs measure $\mu_\mathbf{J}$. Moreover the measure $\mu_\mathbf{J}$ has an exponential decay of correlations in the sense that there is a positive nonrandom constant M such that for any two local functions F and G (i.e., functions dependent only on a finite number of spins) we have

$$|\mu_\mathbf{J}(FG) - \mu_\mathbf{J}(F)\,\mu_\mathbf{J}(G)| \leqslant C(\mathbf{J}, F, G)\, e^{-M\,\mathrm{dist}(F,G)} \tag{1.3}$$

with some positive random variable $C(\mathbf{J}, F, G)$.

In the present paper we consider the corresponding stochastic dynamics defined by the generators

$$\mathscr{L}_{\Lambda,\sigma}^X \equiv \mathscr{L}_{\mathbf{J},\Lambda,\sigma}^X \equiv \delta_{\sigma_{\Lambda^c}} \sum_{X+\mathbf{i} \subset \Lambda} \mathscr{L}_{X+\mathbf{i}} \tag{1.4}$$

with

$$\mathscr{L}_{X+\mathbf{i}} f(\sigma) \equiv \mathscr{L}_{X+\mathbf{i},\mathbf{J}} f(\sigma) \equiv \mu_{X+\mathbf{i},\mathbf{J}}^\sigma f - f(\sigma) \tag{1.5}$$

for some $X \in \mathscr{F}$. Let

$$P_t^{X, \Lambda, \sigma} \equiv P_t^{X, \mathbf{J}, \Lambda, \sigma} \equiv \exp(t\mathscr{L}_{\Lambda, \sigma}^X) \tag{1.6}$$

denote the corresponding semigroup. If $\Lambda = \mathbb{Z}^d$, we will omit the corresponding index. Then we will omit also the index σ, because as one can show, the corresponding dynamics is independent of it. Additionally, to simplify the notation, we will omit the index **J** whenever this will not lead to confusion. Thus we will use the following notation for the infinite-volume Glauber dynamics:

$$P_t^X \equiv P_t^{X, \mathbf{J}} \equiv \exp(t\mathscr{L}_{\mathbf{J}}^X) \equiv \exp(t\mathscr{L}^X)$$

with the corresponding generator

$$\mathscr{L}^X \equiv \sum_{\mathbf{i} \in \mathbb{Z}^d} \mathscr{L}_{X+\mathbf{i}}$$

Let us recall that the generator $\mathscr{L}_{\mathbf{J}}^X$ can be extended in $L_2(\mu_{\mathbf{J}})$ to a self-adjoint (unbounded) operator, which will be denoted later by the same symbol.

We are interested in the ergodicity question for the Glauber dynamics $P_t^{X, \mathbf{J}}$. That is, we want to know what is the best estimate of the quantity $|P_t^{X, \mathbf{J}} f - \mu_{\mathbf{J}} f|$ when time t increases to infinity, which is true with E-probability one.

In the situation of nonrandom spin systems, it is believed that in the uniqueness region the decay to equilibrium for every Glauber dynamics P_t^X is essentially the same as the decay of correlations for the unique Gibbs measure. We will show that this cannot be true for all random spin systems for which (1.3) holds. Namely we will prove the following result.

Theorem 1.1. Suppose that

$$E\{J_{\mathbf{ii'}} \geqslant J\} > 0 \tag{1.7}$$

for all $J \in (0, \infty)$ if $d = 1$, or when $d > 1$ for some sufficiently large $J > 0$. Then with E-probability one, we have

$$\inf_{f \neq 0} \frac{\mu_{\mathbf{J}}(-\mathscr{L}_{\mathbf{J}}^X(f) f)}{\mu_{\mathbf{J}}(f - \mu_{\mathbf{J}} f)^2} = 0 \tag{1.8}$$

i.e., the self-adjoint operator $-\mathscr{L}_{\mathbf{J}}^X$ in $L_2(\mu_{\mathbf{J}})$ has no spectral gap.

The proof of this result (using similar arguments as those employed to get the Griffiths singularities) is given in Section 2. By a simple use of the spectral theorem, this result clearly implies that (when one allows sufficiently large couplings) *the decay to equilibrium cannot be exponentially fast* and therefore it should be qualitatively different than the decay of correlations (1.3).

To answer the question of what could be the decay to equilibrium, we study later in Sections 3 and 4 a simple one-dimensional model with unbounded couplings (otherwise the results of ref. 12 would apply). Using a version of multiscale analysis (which is already quite involved in one dimension), we show the following result.

Theorem 1.2. Suppose that there are constants $\alpha \in (0, 1)$ and $B \in (0, \infty)$ such that for some $z \in (0, \infty)$ we have

$$E \exp(z\, |J_{\mathrm{i},\mathrm{i}+1}|^{1/\alpha}) \leqslant B \tag{1.9}$$

Then for any dynamics $P_t^{X,\mathbf{J}}$ we have *the strong decay to equilibrium*, i.e., the following estimate holds with E-probability one with arbitrary constant $\delta \in (0, 1)$ for any local function f:

$$\|P_t^{X,\mathbf{J}} f - \mu_{\mathbf{J}} f\|_u \leqslant C_\delta(\mathbf{J}, X, f)\, e^{-t^\delta} \tag{1.10}$$

with some random variable $C_\delta(\mathbf{J}, X, f) \in (0, \infty)$.

In view of Theorem 1.1, the above result is the best possible.

The basic ingredients of the proof of Theorem 1.2 are the following:

- *The finite speed of propagation of interaction*, i.e., the following bound: For every $A \in (0, \infty)$ we have

$$\|P_s f - P_s^{\Lambda,\sigma} f\|_u \leqslant e^{-As}\, |||f||| \tag{$*$}$$

for every local function f and every $s \in [0, t]$ with t satisfying

$$\operatorname{dist}(f, \Lambda^c) \geqslant Ct \tag{1.11}$$

for some positive constant C dependent only on the choice of the constant A.

This estimate easily follows by standard arguments (see, e.g., ref. 8), since by our definition the rates of the dynamics are uniformly bounded from above.

- *The logarithmic Sobolev inequality with local coefficients*, i.e., the inequality

$$\mu_\Lambda^\sigma f^2 \log f \leqslant \sum_{\mathrm{i} \in \Lambda} \mathbf{c}_\mathrm{i} \mu_\Lambda^\sigma\, |\nabla_\mathrm{i} f|^2 + \mu_\Lambda^\sigma f^2 \log(\mu_\Lambda^\sigma f^2)^{1/2} \tag{LLS}$$

with some $\mathbf{c_i} \equiv \mathbf{c_i}(\mathbf{J}) \in (0, \infty)$ satisfying (with E-probability one) the following sublinear growth condition:

$$\mathbf{c_i} \leqslant C(\mathbf{J})[1 + d(0, \mathbf{i})]^{\gamma} \tag{**}$$

with some $\gamma \in (0, 1)$ and positive random variable $C(\mathbf{J})$ for every function $f \geqslant 0$.

The proof of (LLS) in our setting is given in Section 3. From the above two properties, the main result follows by use of the ideas of ref. 5 (see also refs. 6, 8, and 9). For the reader's convenience we give the corresponding details in Section 4.

One can hope that the ideas of the present paper will be of use in understanding also the intriguing problem of the decay to equilibrium for disordered spin systems on a higher-dimensional lattice.

2. NO SPECTRAL GAP PROPERTY

Let us consider a spin system on a lattice $\mathbb{Z}^d$, $d \in \mathbb{N}$, described by the interaction

$$U_\Lambda \equiv -\sum_{\mathrm{nn}} J_{\mathbf{ii'}} \sigma_{\mathbf{i}} \sigma_{\mathbf{i'}} \tag{2.1}$$

with summation running over the nearest neighbors and $J_{\mathbf{ii'}}$ being i.i.d. random variables with a distribution E. Let

$$\mathscr{L}_\mathbf{J} = \mathscr{L}_\mathbf{J}^X = \sum_{\mathbf{i} \in \mathbb{Z}^d} \mathscr{L}_{X+\mathbf{i}, \mathbf{J}} \tag{2.2}$$

be the generator of the Glauber dynamics defined in Section 1. Then we have the following result.

Theorem 2.1. Suppose that

$$E\{J_{\mathbf{ii'}} \geqslant J\} > 0 \tag{2.3}$$

for all $J \in (0, \infty)$ if $d = 1$, or when $d > 1$ for some sufficiently large $J > 0$. Then with E-probability one, we have

$$\inf_{f \neq 0} \frac{\mu_\mathbf{J}(-\mathscr{L}_\mathbf{J}^X(f) f)}{\mu_\mathbf{J}(f - \mu_\mathbf{J} f)^2} = 0 \tag{2.4}$$

i.e., the self-adjoint operator $-\mathscr{L}_\mathbf{J}^X$ in $L_2(\mu_\mathbf{J})$ has no spectral gap.

Proof. First let us consider the single-spin-flip case, i.e., dynamics defined with $X=0$. Suppose, for a box $\Lambda \subset \mathbb{Z}^d$ and a positive number J, a configuration $\mathbf{J}$ satisfies

$$\forall(\mathbf{ii}')\subset\Lambda, \qquad J_{\mathbf{ii}'}\geqslant J \qquad \text{and} \qquad \forall \mathbf{i}\in\Lambda, \mathbf{i}'\in\partial\Lambda, \quad |J_{\mathbf{ii}'}|\leqslant 1 \tag{2.5}$$

For such a configuration $\mathbf{J}$ we have

$$\frac{\mu_{\mathbf{J}}(-\mathscr{L}_{\mathbf{J}}(f)f)}{\mu_{\mathbf{J}}(f-\mu_{\mathbf{J}}f)^2}\leqslant Ce^{6\,|\partial\Lambda|}\frac{\mu_{\mathbf{J}}^{\partial\Lambda}(-\mathscr{L}_{\mathbf{J}}^{\partial\Lambda}(f)f)}{\mu_{\mathbf{J}}^{\partial\Lambda}(f-\mu_{\mathbf{J}}^{\partial\Lambda}f)^2} \tag{2.6}$$

where $\mu_{\mathbf{J}}^{\partial\Lambda}$, respectively $\mathscr{L}_{\mathbf{J}}^{\partial\Lambda}$, denotes an infinite-volume measure corresponding to the interaction (2.1) with $\mathbf{J}_{\mathbf{ii}'}\equiv 0$ if $\mathbf{i}\in\Lambda$ and $\mathbf{i}'\in\mathbb{Z}^d\backslash\Lambda$, respectively the corresponding generator of the Glauber dynamics; the constant $C\in(0,\infty)$ is independent of Λ, the configuration $\mathbf{J}$ satisfying (2.5), and a function f. If we now restrict ourselves to the functions f dependent only on spins in Λ, using the result of ref. 10, for $d\geqslant 2$, we get

$$\inf_{\substack{f\neq 0\\ f(\sigma)=f(\sigma_\Lambda)}}\frac{\mu_{\mathbf{J}}^{\partial\Lambda}(-\mathscr{L}_{\mathbf{J}}^{\partial\Lambda}(f)f)}{\mu_{\mathbf{J}}^{\partial\Lambda}(f-\mu_{\mathbf{J}}^{\partial\Lambda}f)^2}=\inf_{\substack{f\neq 0\\ f(\sigma)=f(\sigma_\Lambda)}}\frac{\mu_{\Lambda,\mathbf{J}}(-\mathscr{L}_{\Lambda,\mathbf{J}}(f)f)}{\mu_{\Lambda,\mathbf{J}}(f-\mu_{\Lambda,\mathbf{J}}f)^2}\leqslant C_1(J)e^{-J\alpha^*\,|\partial\Lambda|} \tag{2.7}$$

with some constant $C_1(J)\in(0,\infty)$ and a constant $\alpha^*>0$ independent of J. Combining (2.6) with (2.7) and choosing $J\in(0,\infty)$ sufficiently large, we see that, if Λ is sufficiently large, for every $\varepsilon>0$ we get

$$\inf_{f\neq 0}\frac{\mu_{\mathbf{J}}(-\mathscr{L}_{\mathbf{J}}(f)f)}{\mu_{\mathbf{J}}(f-\mu_{\mathbf{J}}f)^2}<\varepsilon \tag{2.8}$$

To reach a similar conclusion for the one-dimensional case, we use first similar arguments as in (2.6) and extend the measure to an infinite-volume measure with purely ferromagnetic interaction of an amplitude J (outside Λ). Now we take into account the fact that if Λ is sufficiently large, we would not need to make an essential correction when estimating the infimum of interest to us by replacing it by an infimum over all admissible (nonlocalized) functions. Finally, we can estimate from above the spectral gap of the infinite system by the mass gap describing the exponential decay of correlations for the infinite-volume measure. Since the last is converging to zero when J converges to infinity, the former also have to converge to zero.

To finish the proof, we need only to observe that by the Borel–Cantelli lemma for E-a.e. configuration $\mathbf{J}$ there is an arbitrarily large box Λ such that (2.5) is true. This ends the proof of Theorem 2.1 for the single-spin-flip Glauber dynamics.

Let us note that the proof of the general case in one dimension requires no changes, since the finite speed of propagation of interaction on which it is based remains true also for the other dynamics.

To handle the higher-dimensional case, let us remark first that all the arguments go through if we replace the first requirement in (2.5) by

$$J + C > \mathbf{J}_{\mathbf{i}\mathbf{i}'} \geqslant J \tag{2.9}$$

with a constant $C \in [1, \infty)$. In this situation we can bound the quadratic form of a generator $\mathscr{L}_{\mathbf{J}}^{X}$ restricted to the functions in a box Λ by the quadratic form of $\mathscr{L}_{\mathbf{J}}$, the generator of the single-spin-flip dynamics, multiplied by a factor $e^{a(J+C)|X|}$, with some numerical constant $a > 0$. Certainly this factor can be beaten by the one from the right-hand side of (2.7) for all sufficiently large boxes Λ. Therefore also in the general case the conclusion of the theorem remains valid. ∎

To understand better our result, let us remark that by use of the spectral theorem we have the equivalence

$$m \equiv \text{gap}_2(-\mathscr{L}) > 0 \Leftrightarrow \mu(P_t f - \mu f)^2 \leqslant e^{-2mt} \mu(f - \mu f)^2 \qquad \forall f \in L_2(\mu) \tag{2.10}$$

where $\text{gap}_2(-\mathscr{L})$ denotes the spectral gap of minus a self-adjoint generator $\mathscr{L}$ of the Markov semigroup $P_t \equiv e^{t\mathscr{L}}$. Theorem 2.1 does not say that the estimate such as that on the right-hand side of (2.10) is not possible for a given function f. (Of course, by abstract arguments, there are functions for which it is true, although one would not expect that for the local functions.) It does say only that in general to get a bound on the quantity $\mu(P_t f - \mu f)^2$ we need essentially to know the spectral measure of f associated to the self-adjoint operator $\mathscr{L}$. This is a very interesting and challenging problem. A technique for solving it is slowly emerging.

In the rest of the paper we show that in the case of one-dimensional systems, one can obtain more detailed information about the decay to equilibrium.

3. PROOF OF (LLS)

The proof of (LLS) will be obtained by some modification of the arguments from ref. 12. First of all let us note the following simple probabilistic lemma, which will play an important role later.

Lemma 3.1. Suppose that there are constants $\alpha \in (0, 1)$ and $B \in (0, \infty)$ such that for some $z \in (0, \infty)$ we have

$$E \exp(z\,|J_{\mathbf{i},\mathbf{i}+1}|^{1/\alpha}) \leqslant B \tag{3.1}$$

Then the following set has E-measure one:

$$\mathbb{J} \equiv \bigcup_{\substack{a \in \mathbb{N} \\ N \in \mathbb{N}}} \{\mathbf{J}: \forall \mathbf{i},\ |\mathbf{i}| \geqslant N,\ |J_{\mathbf{i},\mathbf{i}+1}| \leqslant a(\log(|\mathbf{i}|+2))^{\alpha}\} \tag{3.2}$$

Proof. Let us observe that under our conditions, for any $N \in \mathbb{N}$ and $a \in \mathbb{N}$, we have

$$\begin{aligned} &E\{\exists \mathbf{i},\ |\mathbf{i}| \geqslant N: |J_{\mathbf{i},\mathbf{i}+1}| \geqslant a(\log(|\mathbf{i}|+2))^{\alpha}\} \\ &\quad \leqslant \sum_{|\mathbf{i}| \geqslant N} E\{|J_{\mathbf{i},\mathbf{i}+1}| \geqslant a(\log(|\mathbf{i}|+2))^{\alpha}\} \\ &\quad = \sum_{|\mathbf{i}| \geqslant N} E\{\exp(z\,|J_{\mathbf{i},\mathbf{i}+1}|^{1/\alpha}) \geqslant \exp[z a^{1/\alpha} \log(|\mathbf{i}|+2)]\} \\ &\quad \leqslant E \exp(z\,|J_{\mathbf{i},\mathbf{i}+1}|^{1/\alpha}) \sum_{|\mathbf{i}| \geqslant N} \exp[-z a^{1/\alpha} \log(|\mathbf{i}|+2)] \end{aligned} \tag{3.3}$$

Hence we see that the rhs of (3.3) can be made arbitrarily small by choosing a such that $a^{1/\alpha} z > 1$ and N sufficiently large. Thus the union of complements of the sets considered on the lhs of (3.3) has E-measure one. ∎

Let us now choose a sequence of lengths $\{D_k \in \mathbb{N}\}_{k \in \mathbb{Z}^+}$ by setting $D_0 \equiv 0$ and

$$\begin{aligned} D_1 &\equiv L > 2 \\ D_{k+1} &\equiv D_k + [D_k^{\delta}] \\ D_{-k} &\equiv D_k \end{aligned} \tag{3.4}$$

with some constants $L \equiv L(\mathbf{J})$ sufficiently large and $\delta \in (0, 1)$ sufficiently small to be chosen later; the symbol $[x]$ denotes the biggest least integer of the real number x. For later purposes, let us note that by our definition we have

$$D_{k+1} - D_k = [D_k^{\delta}] \leqslant [D_{k+1}^{\delta}] \leqslant D_{k+1}^{\delta} \tag{3.5}$$

Using the just introduced sequence of scales, we define the following two families of intervals $\{\Lambda_n^{(l)}\}_{n \in \mathbb{Z}}$, $l = 0, 1$, by setting

$$\Lambda_n^{(l)} \equiv [D_{2n+l} + 1, D_{2(n+1)+l} - 1] \tag{3.6}$$

To proceed further, we will need to prove first a crude estimate, given in Lemma 3.3, on the growth of log-Sobolev coefficients for finite-volume Gibbs measures. In its proof we will use the log-Sobolev inequality for the free measure and the following simple fact.

Lemma 3.2 *(Bar mass gap).* For any $\Lambda \in \mathscr{F}$ we have

$$m \cdot \mu_\Lambda^\sigma (f - \mu_\Lambda^\sigma f)^2 \leqslant \mu_\Lambda^\sigma |\nabla_\Lambda f|^2 \equiv \mu_\Lambda^\sigma \sum_{\mathbf{i} \in \Lambda} |\nabla_{\mathbf{i}} f|^2 \tag{3.7}$$

with *a mass gap* $m \equiv m(\mathbf{J}, \Lambda)$ satisfying

$$m \geqslant \tfrac{1}{2} |\Lambda|^{-1} e^{-4 \sup_{\mathbf{i} \in \Lambda} \|U_{\mathbf{i}}\|_u} \tag{3.8}$$

In particular, for every $\mathbf{J} \in \mathbb{J}$ we have

$$\begin{aligned} m(\mathbf{J}, \Lambda_n^{(l)}) &\geqslant \tfrac{1}{2}(D_{2(n+1)+l} + 1)^{-\delta} \exp\{-4a(\mathbf{J})(\log(D_{2(n+1)+l} + 2))^\alpha\} \\ &\geqslant \tilde{a}(\mathbf{J})(D_{2(n+1)+l} + 1)^{-\delta'} \end{aligned} \tag{3.9}$$

with some positive constant $\tilde{a}(\mathbf{J})$ and any $\delta' \in (\delta, 1)$.

Proof. The proof uses simple "cutting-and-pasting" arguments (which can be applied in any dimensions). For $\Lambda \in \mathscr{F}$, let $\{\mathbf{i}_k : k = 1,..., |\Lambda|\}$ be a natural ordering of its elements. (In higher dimensions one has to replace it by a suitably chosen lexicographic order.) Now for any two configurations $\sigma, \tilde{\sigma} \in \mathbf{M}^{\mathbb{Z}}$ we define an interpolating sequence $\{\sigma^{(k)} \in \mathbf{M}^{\mathbb{Z}}\}_{k \in \mathbb{Z}^+}$ by declaring $\sigma^{(0)} \equiv \sigma$ and setting

$$\sigma_{\mathbf{j}}^{(k)} \equiv \begin{cases} \sigma_{\mathbf{j}} & \text{if } \mathbf{j} \leqslant \mathbf{i}_{k-1} \\ \tilde{\sigma}_{\mathbf{j}} & \text{if } \mathbf{j} \geqslant \mathbf{i}_k \end{cases} \tag{3.10}$$

With this notation we have

$$\begin{aligned} \mu_\Lambda^\sigma (f - \mu_\Lambda^\sigma f)^2 &= \frac{1}{2} \mu_\Lambda^\sigma \otimes \tilde{\mu}_\Lambda^\sigma (f(\sigma) - f(\tilde{\sigma}))^2 \\ &= \frac{1}{2} |\Lambda|^2 \mu_\Lambda^\sigma \otimes \tilde{\mu}_\Lambda^\sigma \left\{ \frac{1}{|\Lambda|} \sum_{k=1,..., |\Lambda|} (f(\sigma^{(k)} - f(\sigma^{(k-1)})) \right\}^2 \end{aligned} \tag{3.11}$$

where σ, respectively $\tilde{\sigma}$, denotes the integration variable with respect to the measure μ_Λ^σ, respectively $\tilde{\mu}_\Lambda^\sigma$. Hence by the Hölder inequality we get

$$\mu_\Lambda^\sigma (f - \mu_\Lambda^\sigma f)^2 \leqslant \tfrac{1}{2} |\Lambda| \sum_{k=1,..., |\Lambda|} \mu_\Lambda^\sigma \otimes \tilde{\mu}_\Lambda^\sigma (f(\sigma^{(k)}) - f(\sigma^{(k-1)}))^2 \tag{3.12}$$

Now we note that

$$\begin{aligned} &\mu_\Lambda^\sigma \otimes \tilde{\mu}_\Lambda^\sigma (f(\sigma^{(k)}) - f(\sigma^{(k-1)}))^2 \\ &\quad \leqslant 2\mu_\Lambda^\sigma \otimes \tilde{\mu}_\Lambda^\sigma |\nabla_{\mathbf{i}_k} f(\sigma^{(k)})|^2 + 2\mu_\Lambda^\sigma \otimes \tilde{\mu}_\Lambda^\sigma |\nabla_{\mathbf{i}_k} f(\tilde{\sigma}^{(k)})|^2 \end{aligned} \tag{3.13}$$

Removing the interaction at the point $\mathbf{i}_k$ (or, if we are at higher dimensions, on the corresponding hyperplane containing this point) from each term on the rhs of (3.13), we "cut" the corresponding integrations and replaced them by integrations with suitable product measures. Next, inserting suitable interaction, we can "paste" the left piece coming from the one measure to the right piece coming from the other measure. By this argument we get the inequality

$$\mu_\Lambda^\sigma \otimes \tilde{\mu}_\Lambda^\sigma (f(\sigma^{(k)}) - f(\sigma^{(k-1)}))^2 \leqslant 4e^{4\,\|U_{\mathbf{i}_k}\|_u} \mu_\Lambda^\sigma\, |\nabla_{\mathbf{i}_k} f(\sigma^{(k)})|^2 \tag{3.14}$$

Combining this with (3.10)–(3.13), we arrive to the mass gap inequality (3.7) with the spectral gap satisfying (3.8). Now the second part of the lemma easily follows by taking into the account Lemma 3.1, the definition of interaction, together with the definition of the intervals $\Lambda_n^{(l)}$ given before and the following simple inequality true whenever $\alpha \in (0, 1)$:

$$\exp\{4a(\mathbf{J})(\log(D_{2(n+1)+l} + 2))^\alpha\} \leqslant \tilde{a}(\mathbf{J})(D_{2(n+1)+l} + 1)^{\delta' - \delta} \tag{3.15}$$

with any $\delta' > \delta$ and some constant $\tilde{a}$ dependent only on $a(\mathbf{J})$ and the choice of δ'. ∎

Lemma 3.3 *(Bar log-Sobolev coefficients).* For any $\Lambda \in \mathscr{F}$ we have

$$\mu_\Lambda^\sigma f^2 \log f \leqslant c\mu_\Lambda^\sigma\, |\nabla_\Lambda f|^2 + \mu_\Lambda^\sigma f^2 \log(\mu_\Lambda^\sigma f^2)^{1/2} \tag{3.16}$$

with *a log-Sobolev coefficient* $c \equiv c(\mathbf{J}, \Lambda)$ satisfying

$$c \leqslant c_1\, |\Lambda|^2\, e^{10 \sup_{\mathbf{i} \in \Lambda} \|U_\mathbf{i}\|_u} \tag{3.17}$$

for some positive numerical constant c_1. In particular for every $\mathbf{J} \in \mathbb{J}$ we have

$$\begin{aligned} c(\mathbf{J}, \Lambda_n^{(l)}) &\leqslant \tilde{c}(D_{2(n+1)+l} + 1)^{+2\delta} \exp\{+6a(\mathbf{J})(\log(D_{2(n+1)+l} + 2))^\alpha\} \\ &\leqslant c_0(\mathbf{J})(D_{2(n+1)+l} + 1)^{+\delta'} \end{aligned} \tag{3.18}$$

with some positive constants $\tilde{c}$ and c_0 dependent on $\mathbf{J}$, and any $\delta' \in (2\delta, 1)$.

Proof. Applying the logarithmic Sobolev inequality for the free measure μ_0 with the function

$$\left(\frac{d\mu^\sigma_{\Lambda|\Sigma_\Lambda}}{d\mu_{0|\Sigma_\Lambda}}\right)^{1/2} f$$

it is not very difficult to see that the following inequality is true:

$$\begin{aligned} \mu_\Lambda^\sigma f^2 \log f &\leqslant C_1 e^{4 \max_{\mathbf{j} \in \Lambda} \|U_\mathbf{j}\|_u} \mu_\Lambda^\sigma\, |\nabla_\Lambda f|^2 \\ &\quad + C_2\, |\Lambda|\, e^{6 \max_{\mathbf{j} \in \Lambda} \|U_\mathbf{j}\|_u} \mu_\Lambda^\sigma\, |f|^2 + \mu_\Lambda^\sigma f^2 \log(\mu_\Lambda^\sigma f^2)^{1/2} \end{aligned} \tag{3.19}$$

Using this, by standard arguments[2] one gets a logarithmic Sobolev inequality for the measure μ_Λ^σ with corresponding coefficient c satisfying

$$c = C_1 e^{4 \max_{i \in \Lambda} \|U_i\|_u} + \frac{1}{m} C_2 |\Lambda| e^{6 \max_{j \in \Lambda} \|U_j\|_u}$$

$$\leqslant \max(C_1, C_2)\, e^{6 \max_{j \in \Lambda} \|U_j\|_u} \left(\frac{|\Lambda|}{m} + 1\right) \tag{3.20}$$

with m denoting the corresponding mass gap. Hence, applying Lemma 3.2, we obtain (3.17). The inequality (3.18) in the case when $\mathbf{J} \in \mathbb{J}$ follows from Lemma 3.1, the definition of the sets $\Lambda_n^{(l)}$, and inequality (3.15). ∎

We will need another simple fact, namely the following cluster property of conditional measures.

Lemma 3.4. For any $\mathbf{J} \in \mathbb{J}$ there is $L(\mathbf{J})$ such that for any $n \in \mathbb{Z}$ and $i = 0, 1$ we have

$$\sup_{\sigma_{\mathbb{Z}\setminus\Lambda_n^{(l)}} = \tilde{\sigma}_{\mathbb{Z}\setminus\Lambda_n^{(l)}}} |\mu^\sigma_{\bar{\Lambda}_n^{(l)}}(\nabla_{\mathbf{j}} U_{\Lambda_n^{(l)}}) - \mu^{\tilde{\sigma}}_{\bar{\Lambda}_n^{(l)}}(\nabla_{\mathbf{j}} U_{\Lambda_n^{(l)}})| \leqslant \exp\{-\tilde{M}(D_{2n+i} + 1)^{\delta/4}\} \tag{3.21}$$

where

$$\bar{\Lambda}_n^{(i)} \equiv \Lambda_n^{(i)} \cap \{\Lambda_{n-1}^{(i + _{\mathrm{mod}\,2}\, 1)} \cup \Lambda_n^{(i + _{\mathrm{mod}\,2}\, 1)}\}$$

Proof. To prove (3.21), we note that, for any $\mathbf{j} \in \mathbb{Z}\setminus\Lambda_n^{(l)}$, $d(\mathbf{j}, \Lambda_n^{(l)}) \leqslant 1$, and for any two configurations σ and $\tilde{\sigma}$ coinciding outside a set $\Lambda_k^{(l)}$, we have

$$|\mu^\sigma_{\bar{\Lambda}_n^{(l)}}(\nabla_{\mathbf{j}} U_{\Lambda_n^{(l)}}) - \mu^{\tilde{\sigma}}_{\bar{\Lambda}_n^{(l)}}(\nabla_{\mathbf{j}} U_{\Lambda_n^{(l)}})| = \left|\mu^\sigma_{\bar{\Lambda}_n^{(l)}}\left(\nabla_{\mathbf{j}} U_{\Lambda_n}, \frac{d\mu^{\tilde{\sigma}}_{\bar{\Lambda}_n^{(l)}}}{d\mu^\sigma_{\bar{\Lambda}_n^{(l)}}}\right)\right| \tag{3.22}$$

with $\nabla_{\mathbf{j}} U_{\Lambda_n^{(l)}}$ localized close to the boundary of $\Lambda_n^{(l)}$ and $d\mu^{\tilde{\sigma}}_{\bar{\Lambda}_n^{(l)}}/d\mu^\sigma_{\bar{\Lambda}_n^{(l)}}$ localized far from the boundary. Then it is not very hard to see that the following estimate is true if $\mathbf{j}$ is on the left of $\Lambda_n^{(l)}$ (other cases being similar):

$$(3.22) \leqslant \|\nabla_{\mathbf{j}} U_{\Lambda_n^{(l)}}\|_u \cdot \left\|\frac{d\mu^{\tilde{\sigma}}_{\bar{\Lambda}_n^{(l)}}}{d\mu^\sigma_{\bar{\Lambda}_n^{(l)}}}\right\|_u \prod_{\mathbf{i} \in A(\mathbf{j}, n, l)} \operatorname{th}(2 \|U_{\mathbf{i}}\|_u) \tag{3.23}$$

where we have set

$$A(\mathbf{j}, n, l) \equiv [\mathbf{j} + 1, [(D_{2(n+1)+l} - D_{2n+l} - 2)/2]]$$

Fixing attention on the case $n \geqslant 0$ (the other cases being similar), and taking $\mathbf{J} \in \mathbb{J}$, we get

$$\operatorname{th}(2 \|U_{\mathbf{i}}\|_u) \leqslant 1 - \exp\{-2a(\mathbf{J})(\log(\mathbf{i} - D_{2n+l}))^\alpha\} \tag{3.24}$$

Since for any $\delta'' \in (0, 1)$ we have

$$\exp\{-2a(\mathbf{J})(\log(\mathbf{i}-D_{2n+l}))^{\alpha}\} \geqslant b_1(\mathbf{J})(\mathbf{i}-D_{2n+l})^{-\delta''} \tag{3.25}$$

with some positive constant $b_1(\mathbf{J})$, we get the following estimate:

$$\prod_{\mathbf{i}\in\Lambda(\mathbf{j},n,l)} \operatorname{th}(2\,\|U_{\mathbf{i}}\|_u) \leqslant \exp\{-b_2(D_{2(n+1)+l}^{1-\delta''} - D_{2n+l}^{1-\delta''})\} \tag{3.26}$$

with some positive constant $b_2 \equiv b_2(\mathbf{J})$. Choosing $\delta'' \in (0, \frac{1}{4}\delta)$ and using our assumption about the interaction to estimate the first two factors on the rhs of (3.23), we obtain

$$|\mu^{\sigma}_{\bar{\Lambda}^{(l)}_n}(\nabla_{\mathbf{j}} U_{\Lambda^{(l)}_n}) - \mu^{\tilde{\sigma}}_{\bar{\Lambda}^{(l)}_n}(\nabla_{\mathbf{j}} U_{\Lambda^{(l)}_n})| \leqslant \exp\{-\tilde{M}(D_{2n+l}+1)^{\delta/4}\} \tag{3.27}$$

with some positive constant $\tilde{M} \equiv \tilde{M}(\mathbf{J})$. This ends the proof of the lemma. ∎

Using the above proven lemmas, we will show the following main result of this section.

Theorem 3.5. Suppose the distribution E satisfies the exponential bound (3.1) with some $\alpha \in (0, 1)$. Then there is $\gamma \in (0, 1)$ such that for every $\Lambda \in \mathscr{F}$, $\sigma \in \mathbf{M}^{\mathbb{Z}}$, and $\mathbf{J} \in \mathbb{J}$ the corresponding conditional measure μ^{σ}_{Λ} satisfies the following logarithmic Sobolev inequality with local coefficients:

$$\mu^{\sigma}_{\Lambda} f^2 \log|f| \leqslant \sum_{\mathbf{i}\in\Lambda} \mathbf{c}_{\mathbf{i}} \mu^{\sigma}_{\Lambda} |\nabla_{\mathbf{i}} f|^2 + \mu^{\sigma}_{\Lambda} f^2 \log(\mu^{\sigma}_{\Lambda} f^2)^{1/2} \tag{LLS}$$

where $\mathbf{c}_{\mathbf{i}} \equiv \mathbf{c}_{\mathbf{i}}(\mathbf{J}) \in (0, \infty)$ satisfies (with E-probability one) the sublinear growth condition

$$\mathbf{c}_{\mathbf{i}} \leqslant C(\mathbf{J})\{1 + d(0, \mathbf{i})\}^{\gamma} \tag{3.28}$$

with some positive random variable $C(\mathbf{J})$.

Remark. Let us note that (LLS) implies a similar inequality with $\mu(-\mathscr{L}_{X+\mathbf{i}}(f)f)$ replacing $\mu\,|\nabla_{\mathbf{i}} f|^2$ and new coefficients $\tilde{\mathbf{c}}_{\mathbf{i}} \equiv \tilde{\mathbf{c}}_{\mathbf{i}}(X, \mathbf{J})$ satisfying also a sublinear growth condition.

Proof. First of all it is easy to see that the product measures

$$\mathscr{P}_k f(\sigma) \equiv \bigotimes_{n\in\mathbb{Z}} \mu^{\sigma}_{\Lambda^{(k_{\mathrm{mod}\,2})}_n} f, \qquad k \in \mathbb{N}, \quad k_{\mathrm{mod}\,2} \equiv k \bmod 2 \tag{3.29}$$

satisfies the following log-Sobolev inequality with local coefficients:

$$\mathscr{P}_k f^2 \log f \leqslant \sum_{\mathbf{j}\in\mathbb{Z}} c^{(k)}_{\mathbf{j}} \mathscr{P}_k |\nabla_{\mathbf{j}} f|^2 + \mathscr{P}_k f^2 \log(\mathscr{P}_k f^2)^{1/2} \tag{3.30}$$

where $\mathbf{c}_{\mathbf{j}}^{(k)}$, $k \in \mathbb{N}$, satisfies

$$0 \leqslant \mathbf{c}_{\mathbf{j}}^{(k)} \leqslant \begin{cases} c(\mathbf{J}, \Lambda_n^{(k \bmod 2)}) & \text{if } \mathbf{j} \in \Lambda_n^{(k \bmod 2)} \text{ for some } n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{3.31}$$

One can easily see that, as follows from Lemma 3.3, the local coefficients $\{\mathbf{c}_{\mathbf{j}}^{(k)}\}$ satisfy the sublinear growth condition (3.28) for any $\gamma \in (0, 1)$.

For a nonnegative function f let us define the following sequence of functions:

$$f_k \equiv \begin{cases} f & \text{if } k = 0 \\ (\mathscr{P}_k \cdots \mathscr{P}_1 f^2)^{1/2} & \text{for } k \in \mathbb{N} \end{cases} \tag{3.32}$$

Since for every $k \in \mathbb{N}$, $\mathscr{P}_k$ is a regular conditional expectation associated to the unique infinite-volume Gibbs measure $\mu_{\mathbf{J}}$, we have

$$\begin{aligned} \mu_{\mathbf{J}} f_{k-1}^2 \log f_{k-1} &= \mu_{\mathbf{J}} \mathscr{P}_k f_{k-1}^2 \log f_{k-1} \\ &\leqslant \sum_{\mathbf{i} \in \mathbb{Z}} c_{\mathbf{i}}^{(k)} \mu_{\mathbf{J}} \, |\nabla_{\mathbf{i}} f_{k-1}|^2 + \mu_{\mathbf{J}} f_k^2 \log f_k \end{aligned} \tag{3.33}$$

Summing the inequalities (3.33) over $k = 1, \ldots, N$, we obtain

$$\mu_{\mathbf{J}} f^2 \log f \leqslant \sum_{k=1}^{N} \sum_{\mathbf{i} \in \mathbb{Z}} \mu_{\mathbf{J}} (\mathbf{c}_{\mathbf{i}}^{(k)} \, |\nabla_{\mathbf{i}} f_{k-1}|^2) + \mu_{\mathbf{J}} (f_N \log f_N) \tag{3.34}$$

Let us mention that for $k \geqslant 2$ the corresponding function f_{k-1} is localized in the set (of length $2R$)

$$\bar{\Lambda}_n^{(k)} \equiv \Lambda_n^{(k \bmod 2)} \cap \{ \Lambda_{n-1}^{(k + \bmod 2 \, 1)} \cup \Lambda_n^{(k + \bmod 2 \, 1)} \} \tag{3.35}$$

Therefore the corresponding local coefficients $c_{\mathbf{i}}^{(k)}$ are determined by the corresponding log-Sobolev coefficients of the relativizations of the measures

$$\mu^{\sigma}_{\Lambda_n^{(k)}} \, | \Sigma(\bar{\Lambda}_n^{(k)})$$

respectively. Using this remark, it is easy to see that for $k > 1$ we need to take

$$\mathbf{c}_{\mathbf{j}}^{(k)} \equiv \begin{cases} \exp\{2 \, \| U(\bar{\Lambda}_0^{(k \bmod 2)}) \|_u & \text{if } \mathbf{j} \in \bar{\Lambda}_n^{(k \bmod 2)} \text{ for some } n \in \mathbb{Z} \\ 0 . & \text{otherwise} \end{cases} \tag{3.36}$$

Let us note that due to Lemma 3.1, the nonzero terms on the right-hand side of (3.36) satisfy the mild sublinear growth condition [of the form (3.28) with $\gamma > 0$ arbitrarily close to zero].

Remark. Let us note at this point that in the case considered in ref. 12 the corresponding coefficients are bounded by a constant independent of the size of the intervals Λ_n (which can be seen, e.g., by using Lemma 5.1 of ref. 5) and therefore Lemma 1.1 is superfluous there.

Now we use analogous arguments as in the proof of Lemmas 1.2 and 1.3 of ref. 12, but now taking into the account our choice (3.4) of the intervals $\Lambda_n^{(l)}$ with sufficiently large $L \equiv L(\mathbf{J})$ which allows us to apply Lemmas 3.1–3.4. By this we obtain the following bound for $k \geqslant 2$:

$$\sum_{\mathbf{i} \in \mathbb{Z}} \mu_{\mathbf{J}}(\mathbf{c}_{\mathbf{i}}^{(k)} |\nabla_{\mathbf{i}} f_{k-1}|^2) \leqslant \lambda^{k-1} \sum_{\mathbf{i} \in \mathbb{Z}} \mu_{\mathbf{J}}(\mathbf{c}_{\mathbf{i}}^{(1)}(1 + C_0 \lambda^{-1}) |\nabla_{\mathbf{i}} f|^2) \quad (3.37)$$

with some constant $\lambda \in (0, 1)$ and $C_0 \in (0, \infty)$ independent of $\mathbf{J}$ [provided $L(\mathbf{J})$ is chosen sufficiently large]. By similar arguments as in ref. 12 it is not difficult to see that in our situation we have also

$$\lim_{N \to \infty} f_N = \mu_{\mathbf{J}} f \quad (3.38)$$

for any nonnegative local function f. Combining (3.34) with (3.37) and (3.38), we conclude that LLS is true with the local coefficients satisfying the mild sublinear growth condition (3.28). This ends the proof of Theorem 3.5. ∎

Let us remark that clearly also (with probability one) the infinite-volume Gibbs measures $\mu_{\mathbf{J}}$ satisfy the logarithmic Sobolev inequality with corresponding local coefficients satisfying the sublinear growth condition.

4. DECAY TO EQUILIBRIUM

First of all we observe that using the finite speed of propagation property (∗) and the known property of approximation of the infinite-volume Gibbs measure and assuming that the function f of interest to us is localized in a set Λ but far from its boundary, we have

$$\begin{aligned} |P_t f(\sigma) - \mu_{\mathbf{J}} f| &\leqslant |P_t f(\sigma) - P_t^{\Lambda,\sigma} f(\sigma)| + |P_t^{\Lambda,\sigma} f(\sigma) - \mu_\Lambda^\sigma f| + |\mu_\Lambda^\sigma f - \mu_{\mathbf{J}} f| \\ &\leqslant |P_t^{\Lambda,\sigma} f(\sigma) - \mu_\Lambda^\sigma f| + B(\mathbf{J}, f) e^{-At} \end{aligned} \quad (4.1)$$

for some constant $A \in (0, \infty)$ provided $d(f, \partial\Lambda) \geqslant Ct$ with some positive constant C; the constant $B(\mathbf{J}, f) \in (0, \infty)$ is independent of Λ, σ, and t. Next we observe that for any $q \in [2, \infty)$ we have

$$\begin{aligned} |P_t^{\Lambda,\sigma} f(\sigma) - \mu_\Lambda^\sigma f| &= \{ |P_t^{\Lambda,\sigma} f(\sigma) - \mu_\Lambda^\sigma f|^q \}^{1/q} \\ &\leqslant e^{2 \|U_\Lambda\|_u / q} \{ \mu_\Lambda^\sigma |P_t^{\Lambda,\sigma} f(\cdot) - \mu_\Lambda^\sigma f|^q \}^{1/q} \end{aligned} \quad (4.2)$$

where in the last step we have used that the probability of every configuration inside Λ measured with the measure μ_Λ^σ is not smaller than $\exp(-2\,\|U_\Lambda\|_u)$. (Let us note that for continuous spins similar arguments work at the cost of decreasing the time by one.) By our estimates of Section 3, we get the following hypercontractivity estimate:

$$\{\mu_\Lambda^\sigma\,|P_t^{\Lambda,\sigma}f(\cdot)-\mu_\Lambda^\sigma f|^q\}^{1/q}\leqslant\{\mu_\Lambda^\sigma\,|P_{\theta t}^{\Lambda,\sigma}f(\cdot)-\mu_\Lambda^\sigma f|^2\}^{1/2} \tag{4.3}$$

for any q satisfying

$$2\leqslant q\leqslant q(t)\equiv 1+e^{(1-\theta)t/\mathbf{c}_\Lambda} \tag{4.4}$$

with any $\theta\in(0,1)$ and

$$\mathbf{c}_\Lambda=\max_{\mathbf{i}\in\Lambda}\tilde{\mathbf{c}}_\mathbf{i}\leqslant\mathbf{c}(\mathbf{J})\,|\Lambda|^\gamma \tag{4.5}$$

for any $\gamma\in(0,1)$. Using this together with the fact that by general arguments (see, e.g., ref. 2) (4.5) implies a spectral gap

$$\text{gap}_2(-\mathscr{L}_\Lambda^\sigma)\geqslant\frac{1}{\mathbf{c}_\Lambda} \tag{4.6}$$

we arrive at the following bound:

$$\{\mu_\Lambda^\sigma\,|P_t^{\Lambda,\sigma}f(\cdot)-\mu_\Lambda^\sigma f|^q\}^{1/q}\leqslant e^{-\theta t/\mathbf{c}_\Lambda}(\mu_\Lambda^\sigma\,|f-\mu_\Lambda^\sigma f|^2)^{1/2} \tag{4.7}$$

Since we need to take $|\Lambda|=[Ct]$ [to have (4.1)], the estimate (4.5) together with (4.7) and some simple arguments give us

$$\{\mu_\Lambda^\sigma\,|P_t^{\Lambda,\sigma}f(\cdot)-\mu_\Lambda^\sigma f|^q\}^{1/q}\leqslant D_\delta(\mathbf{J},\theta)\,e^{-t^\delta}\,|||f||| \tag{4.8}$$

with some constants $D_\delta(\mathbf{J},\theta)\in(0,\infty)$ and any $\delta<1-\gamma$. Using this, the bounds (4.1)–(4.2), and observing that with probability one

$$\frac{\|U_\Lambda\|_u}{q(t)}\xrightarrow[t\to\infty]{}0 \tag{4.9}$$

we finally arrive at the desired estimate

$$|P_tf(\sigma)-\mu_\mathbf{J}f|\leqslant C_\delta(\mathbf{J},f)e^{-t^\delta} \tag{4.10}$$

with some constant $C_\delta(\mathbf{J},f)\in(0,\infty)$ and $\delta<1-\gamma$. This ends the proof of Theorem 1.2. ▮

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